Post-Processing of A-φ Algorithm for Evaluating Eddy Current Density in Three-Dimensional FEM

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After the magnetic vector potential A and the electric scalar potential φ are obtained by the FEM, as the post-processing, the evaluation of electric field intensity E or eddy current density J is an important step, which influences the accuracy of the simulation greatly. Theoretically, E is the sum of the gradient of φ and the time-differential of A. However, based on the discrete solution of the FEM, how to evaluate the sum of those two terms concerns methodological issue, for the time-differential of A must be nodal values, but the gradient of φ may be element value. The sum of two terms on different positions may create large error. A method for evaluating nodal gradient of φ is presented in this paper, to fulfill that the nodal E and J are evaluated by nodal gradient of φ and nodal time-differential of A. The presented method is a global evaluation method that evaluates the nodal gradient of φ based on the nodal φ in the whole field region by setting and solving another equation system of FEM. Certainly, this post-processing method for evaluating E and J consumes some computing time, but the evaluation accuracy can be increased indeed.

***Index Terms*—Computational electromagnetics, eddy current evaluation, finite element method, gradient evaluation.**

I. Introduction

THE nodal finite element method with $\boldsymbol{A}$-φ potential functions has been widely used to solve 3-d eddy current problems. Generally, high accurate solution of the potentials can be made. However, the accuracy of electric field intensity or eddy current density evaluated from the known potentials may be much lower.

Theoretically, the electric field intensity $\boldsymbol{E}$ is the sum of the gradient of electric scalar potential φ and the time-differential of magnetic vector potential $\boldsymbol{A}$. In the FEM solution, if the field $\boldsymbol{E}$ at a node is to be evaluated, the gradient of φ and the time-differential of $\boldsymbol{A}$ exactly at that node should be known. However, there is no a unique theoretical method to obtain the nodal gradient of φ from the nodal φ. Existing different methods provide different evaluation accuracy and take different computing time [1-2], which evaluate the nodal gradient of φ based on the known φ at nodes around the target node, so that they are local region methods.

A method for evaluating nodal gradient of φ is presented in this paper. It is a global evaluation method that evaluates the nodal gradient of φ based on the nodal φ in the whole field region by setting and solving an equation system of FEM. Therefore, the method is of theoretical completeness. Although it needs extra computing time to solve another equation system of FEM besides the FEM for solving potential $\boldsymbol{A}$-φ, the method can provide high accurate result for $\boldsymbol{E}$ and $\boldsymbol{J}$, which is proved in the solution of transient problems.

II. Global Evaluation Method of Gradient

Based on nodal potential values obtained by the FEM, the eddy current density at any node in conductor could be expressed as:

$$\boldsymbol{J}(\boldsymbol{r}) = -\sigma\nabla\varphi(\boldsymbol{r}) - \sigma\frac{\partial \boldsymbol{A}(\boldsymbol{r})}{\partial t} \tag{1}$$

where φ is electric potential, $\boldsymbol{A}$ is the magnetic vector potential, and σ is the conductivity. It is easy to get the nodal time-differential of A, i.e. the second term of (1), whose accuracy depends on the length of time step and the mesh. But the evaluation of nodal gradient of φ mainly depends on methods. A new method is introduced in the following.

A. Gradient Equation

In order to evaluate the nodal gradient of φ based on known nodal φ, we set an equation first as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r}) \tag{2}$$

where, $\boldsymbol{E}(\boldsymbol{r})$ is regarded as unknown function to be solved, and the right hand side is known. However, we must avoid the evaluation of $\nabla\varphi$, since the solution of gradient will increase or enlarge the error of φ. The FEM will be used to solve (2), which employs the Schaubert-Wilton-Glisson (SWG) basis function [3]. In forming the system of FEM equations, the evaluation of $\nabla\varphi$ will be avoided.

B. Derivation of the FEM Equation System

The weighted residual method is adopted to form the system of FEM equations for (2). By employing the SWG basis function, $\boldsymbol{E}(\boldsymbol{r})$ in (2) can be expressed as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\sum_{n=1}^{N}\boldsymbol{f}_n(\boldsymbol{r})E_n \tag{3}$$

where, N is the number of element faces, E_n represents the normal components of $\boldsymbol{E}(\boldsymbol{r})$ at the nth face, and $\boldsymbol{f}_n(\boldsymbol{r})$ is the SWG basis function [3].

By adopting the SWG basis function as the weighting function, taking the inner product of $\boldsymbol{f}(\boldsymbol{r})$ with (2) yields

$$\int_V \boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})dV = -\int_V \boldsymbol{f}(\boldsymbol{r})\cdot\nabla\varphi(\boldsymbol{r})dV \tag{4}$$

To remove $\nabla\varphi$ in (4), by the divergence theorem and vector calculus [4], (5) could be written as:

$$\int_V \boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})dV = -\int_S \varphi(\boldsymbol{r})\,\boldsymbol{f}(\boldsymbol{r})\cdot\boldsymbol{n}dS + \int_V \varphi(\boldsymbol{r})\nabla\cdot\boldsymbol{f}(\boldsymbol{r})dV \quad (5)$$

With (3) and (5), the finite element equations could be obtained as follows:

$$\sum_{e=1}^{N_e}\sum_{n=1}^{N}\left(\int_{V^e} \boldsymbol{f}_m^e(\boldsymbol{r})\cdot\boldsymbol{f}_n^e(\boldsymbol{r})dV\right)E_n^e = \\ -\sum_{e}^{N_e}\left[\int_{S^e}\varphi^e(\boldsymbol{r})\,\boldsymbol{f}_m^e(\boldsymbol{r})\cdot\boldsymbol{n}dS + \int_{V^e}\varphi^e(\boldsymbol{r})\nabla\cdot\boldsymbol{f}_m^e(\boldsymbol{r})dV\right] \quad (6)$$

where N_e is total number of elements, N is total number of element surfaces, and m from 1 to N expresses the mth equation of equation system. Equation (6) can be written as

$$\mathbf{ZE} = \mathbf{b} \quad (7)$$

According to the expression of the SWG basis function, the element of Z is as follows:

$$Z_{mn} = \frac{a_m a_n}{9V_m^+ V_n^\pm}\int_{T_m^+}\boldsymbol{\rho}_m^+\cdot\boldsymbol{\rho}_n^\pm dV + \frac{a_m a_n}{9V_m^- V_n^\pm}\int_{T_m^-}\boldsymbol{\rho}_m^-\cdot\boldsymbol{\rho}_n^\pm dV \quad (8)$$

In the right hand side of (6), there is the divergence of $\boldsymbol{f}(\boldsymbol{r})$, whose analytical solution can be obtained as follows:

$$\nabla\cdot\boldsymbol{f}_n(\boldsymbol{r}) = \begin{cases} \dfrac{a_n}{V_n^+}, \boldsymbol{r}\in T_n^+ \\ -\dfrac{a_n}{V_n^-}, \boldsymbol{r}\in T_n^- \\ 0, \boldsymbol{r}\notin T_n^\pm \end{cases} \quad (9)$$

Then, the right hand side of (6) can be written as

$$b_m = \begin{cases} \dfrac{a_m}{3V_m^\pm}\displaystyle\int_{a_m}\varphi\boldsymbol{\rho}_m^\pm\cdot\boldsymbol{n}dS \mp \dfrac{a_m}{V_m^\pm}\displaystyle\int_{T_m^\pm}\varphi dV, S_m\in\Gamma \\ \dfrac{a_m}{V_m^-}\displaystyle\int_{T_m^-}\varphi dV - \dfrac{a_m}{V_m^+}\displaystyle\int_{T_m^+}\varphi dV, \qquad S_m\notin\Gamma \end{cases} \quad (10)$$

By solving the finite element equations, the gradient of the potential at nodes can be evaluated. Consequently, the total current density at nodes, which is the combination of the electric potential gradient part and the time variation of magnetic vector potential part, can be obtained in high accuracy.

III. Verification of the Method

In order to verify the accuracy of the global evaluation method, a model with analytical solution is employed, which is a cylinder conductor excited by a harmonic current. The purpose is to compare the numerical solution of the current distribution on the section of the conductor with the analytical solution. Suppose its radius is 0.05m, length is 0.5m, conductivity is 4.41×10^7 S/m, and the frequency of current source is 50 Hz.

Due to the skin effect of eddy current distribution, the current density varies dramatically near conductor surface. Therefore, refining finite element mesh is set near the conductor surface as shown as Fig. 1, which can guarantee both the solution of $\boldsymbol{A}$-φ and the gradient of φ as well as the eddy current. The tetrahedral mesh is employed.

The current density distributions along cylinder radius of the section obtained by analytical solution, traditional method and the global method are given in Fig 2. The traditional method means that the gradient of φ is evaluated in each element by the nodal φ at its four nodes. From the results in Fig 2, we can see that the global method can significantly improve the accuracy of the electric field intensity and the eddy current, compared with the traditional method.

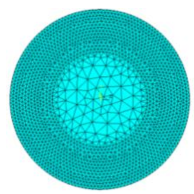

Fig. 1. Mesh on section.

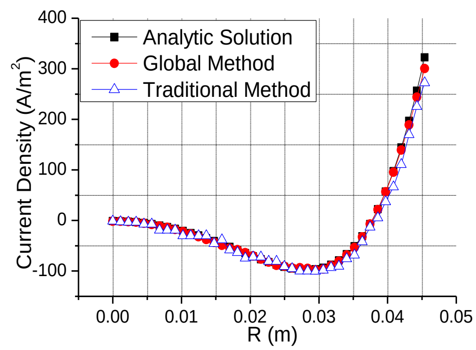

Fig. 2. Current density distributions along radius.

IV. Conclusions

1) The new approach takes account of global electric potential values to evaluate the gradient of the potential by solving a system of FEM equations, so that the nodal gradient can be obtained directly.

2) In forming the FEM stiffness matrix, the computation of electric potential gradient by discrete nodal values can be avoided, which can remove the error of gradient operation, so that the accuracy of global method is improved further.

3) Although the global method is more time consuming, because of solving finite element equations, it can provide high accuracy solution of eddy current and electric field.

4) The new method can not only be used in eddy current evaluation, but also in other gradient evaluation based on known nodal potential values.

V. References

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